# The electrostatic problem of a portion of sphere protruding from a plane electrode in an electric field



1981 J. Phys. A: Math. Gen. 14 945

(http://iopscience.iop.org/0305-4470/14/4/022)

J. Phys. A: Math. Gen. **14** (1981) 945–955. Printed in Great Britain

# The electrostatic problem of a portion of sphere protruding from a plane electrode in an electric field

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Received 9 July 1980, in final form 6 October 1980

**Abstract.** The method used by Lebedev and Skalskaya, for solving the electrostatic problem of a grounded sphere lying on a plane in an electric field, is generalised to the case of a portion of sphere protruding from a plane electrode. Expressions are developed for the electrical field on the surface and along the axis of the sphere. Results are checked in elementary cases for which the solution is already known. Numerical values of charge and force coefficients are tabulated.

## 1. Introduction

Many problems (gas or vacuum discharge initiation, particle levitation, etc) require the knowledge of the field distribution, of the electric charge and of the electrostatic force on a protrusion from a plane electrode.

The solution is more or less readily found by means of elementary computations in simple cases: half-cylinder, hemisphere, half-spheroid, etc (Durand 1966).

In this paper we develop the mathematical treatment of the problem of an electrically stressed conducting sphere more or less embedded in a plane electrode with the method already used by Lebedev and Skalskaya (1962) for a sphere lying on a plane.

## 2. Potential function $V$

Let $a$ be the radius of the sphere and $C$ its centre. The sphere $(C, a)$ and the plane $(z=0)$ intersect in a circle of radius $c$ and centre $O$ (figure 1).

Let $E_0$ be the applied uniform electrical field parallel to the axis $Oz$ of the protrusion. The potential function $V$ is determined by integration of Laplace's equation

$$\Delta V = 0$$

with boundary conditions

$$V|_{z=0}=0 \qquad V|_{\mathscr{S}}=0 \qquad V|_{z\to\infty}=E_0 z.$$

By introducing toroidal coordinates $(\alpha, \beta)$ related to cylindrical coordinates $(r, z)$ by

$$z+\mathrm{i}r=\mathrm{i}c\coth[\tfrac{1}{2}(\alpha+\mathrm{i}\beta)]$$

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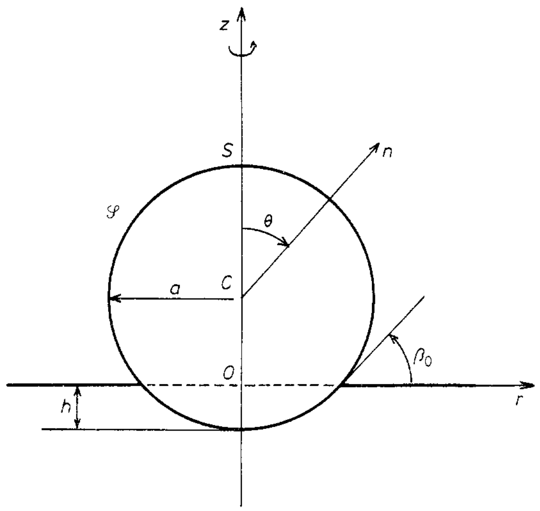


**Figure 1.** Sphere embedded in a plane.

Laplace's equation becomes

$$\frac{\partial}{\partial \alpha}\left(\frac{\sinh \alpha}{\cosh \alpha - \cos \beta}\frac{\partial V}{\partial \alpha}\right)+\frac{\partial}{\partial \beta}\left(\frac{\sinh \alpha}{\cosh \alpha - \cos \beta}\frac{\partial V}{\partial \beta}\right)=0$$

with boundary conditions

$$V|_{\beta=0}=0 \qquad V|_{\beta=\beta_0}=0 \qquad V|_{z\to\infty}=E_0 c\frac{\sin \beta}{\cosh \alpha - \cos \beta}$$

where $\beta_0$ is the value of the $\beta$ coordinate corresponding to the surface $\mathscr{S}$ of the spherical protrusion and defined by $\sin \beta_0 = c/a$.

Substituting $V=(2\cosh\alpha-2\cos\beta)^{1/2}v$, we find a particular solution

$$V_\nu=(2\cosh\alpha-2\cos\beta)^{1/2}(A_\nu P_{\nu-1/2}(\cosh\alpha)+B_\nu Q_{\nu-1/2}(\cosh\alpha))$$
$$\times(C_\nu\cos\nu\beta+D_\nu\sin\nu\beta).$$

$P_{\nu-1/2}(\cosh\alpha)$ and $Q_{\nu-1/2}(\cosh\alpha)$ are Legendre's functions of the first and second kind (Lebedev 1965); $A_\nu$, $B_\nu$, $C_\nu$ and $D_\nu$ are unknown coefficients.

The potential $V$ is finite, and the problem can be solved by superposition of solutions of the form

$$V_\tau=(2\cosh\alpha-2\cos\beta)^{1/2}(M_\tau\cosh\tau\beta+N_\tau\sinh\tau\beta)P_{-1/2+i\tau}(\cosh\alpha)$$

after setting $\nu=i\tau$ ($\tau\geqslant 0$). We look for a solution of the form

$$V=E_0 z+\int V_\tau\,d\tau.$$

Then

$$V=E_0 z+(2\cosh\alpha-2\cos\beta)^{1/2}\int_0^\infty(M_\tau\cosh\tau\beta+N_\tau\sinh\tau\beta)P_{-1/2+i\tau}(\cosh\alpha)\,d\tau$$

but $V = 0$ on the plane $\beta = 0$, hence $M_\tau = 0$ and

$$V = \frac{E_0 c \sin\beta}{\cosh\alpha - \cos\beta} + (2\cosh\alpha - 2\cos\beta)^{1/2} \int_0^\infty (N_\tau \sinh\tau\beta) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau.$$

On the portion of sphere ($\beta = \beta_0$), $V = 0$, hence

$$\int_0^\infty (N_\tau \sinh\tau\beta_0) P_{-1/2+i\tau}(\cosh\alpha)\, d\tau = -2E_0 c \frac{\sin\beta_0}{(2\cosh\alpha - 2\cos\beta_0)^{3/2}}.$$

By differentiation of the well known formula (Lebedev 1965)

$$\frac{1}{(2\cosh\alpha - 2\cos\beta_0)^{1/2}} = \int_0^\infty \frac{\cosh(\pi - \beta_0)\tau}{\cosh\pi\tau} P_{-1/2+i\tau}(\cosh\alpha)\, d\tau$$

we get

$$N_\tau \sinh\tau\beta_0 = -2E_0 c[\tau \sinh(\pi - \beta_0)\tau]/\cosh\pi\tau$$

and after some transformations, we find

$$V(\alpha, \beta) = 2E_0 c(2\cosh\alpha - 2\cos\beta)^{1/2} \int_0^\infty \tau \frac{\sinh\pi\tau}{\cosh\pi\tau} \frac{\sinh(\beta_0 - \beta)\tau}{\sinh\beta_0\tau} P_{-1/2+i\tau}(\cosh\alpha)\, d\tau.$$

$P_{-1/2+i\tau}(\cosh\alpha)$ is sometimes called the conical function or Mehler function (Robin 1958).

## 3. Electrical field $E_{\mathscr{S}}$ on the surface of the portion of sphere

### *3.1. Expression of $E_{\mathscr{S}}/E_0$*

From $V$ and

$$\frac{\partial V}{\partial n} = \frac{\cosh\alpha - \cos\beta}{c} \frac{\partial V}{\partial\beta}$$

we find that

$$\frac{E_{\mathscr{S}}}{E_0} = (2\cosh\alpha - 2\cos\beta_0)^{3/2} \int_0^\infty \frac{\tau^2 \tanh\pi\tau}{\sinh\beta_0\tau} P_{-1/2+i\tau}(\cosh\alpha)\, d\tau.$$

### *3.2. Field $E_S$ on the sphere apex*

In this case $\alpha = 0$ and $P_{-1/2+i\tau}(1) = 1$, so that

$$\frac{E_S}{E_0} = \left(\frac{\sin\frac{1}{2}\beta_0}{\frac{1}{2}\beta_0}\right)^3 \int_0^\infty \frac{x^2 \tanh(\pi x/\beta_0)}{\sinh x}\, dx$$

with

$$\int_0^\infty \frac{x^2 \tanh(\pi x/\beta_0)}{\sinh x}\, dx = \tfrac{7}{2}\zeta(3) - \sum_{n=1}^{\infty} (-1)^{n+1} \zeta(3, \tfrac{1}{2} + \pi n/\beta_0).$$

$\zeta(\alpha, a)$ is the Riemann function (Campbell 1966).

Particular cases:

$\beta_0 = 0$ (sphere)

$$\frac{E_S}{E_0} = \int_0^\infty \frac{x^2}{\sinh x}\,\mathrm{d}x = \tfrac{7}{2}\zeta(3) = 4.207\,199\ldots$$

$\beta_0 = \pi/2$ (hemisphere)

$$\frac{E_S}{E_0} = \left(\frac{4}{\pi}\right)^3 \frac{1}{\sqrt{2}} \int_0^\infty \frac{x^2 \cosh x}{\cosh 2x}\,\mathrm{d}x = 3$$

$\beta_0 = \pi$ (plane)

$$\frac{E_S}{E_0} = \left(\frac{2}{\pi}\right)^3 \int_0^\infty \frac{x^2}{\cosh x}\,\mathrm{d}x = 1.$$

Thus the above results fit well known values.

### 3.3. *Other expressions of $E_{\mathscr{S}}/E_0$*

Substituting the Mehler function by its integral representation (Lebedev *et al* 1965)

$$P_{-1/2+\mathrm{i}\tau}(\cosh\alpha) = \frac{2}{\pi}(\coth \pi\tau) \lim_{x\to\alpha} \int_x^\infty \frac{(\sin x\tau)\,\mathrm{d}x}{(2\cosh x - 2\cosh\alpha)^{1/2}},$$

reversing the order of integration and using the familiar properties of Fourier sine transforms (Erdelyi *et al* 1954), we obtain

$$\int_0^\infty \tau^2 \frac{\tanh \pi\tau}{\sinh \beta_0\tau} P_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau = I$$

$$= \frac{\pi^2}{\beta_0^3} \lim_{x\to\alpha} \int_x^\infty \frac{\mathrm{d}x}{(2\cosh x - 2\cosh\alpha)^{1/2}} \frac{\sinh(\pi x/2\beta_0)}{2\cosh(\pi x/2\beta_0)}$$

and with the substitution $\sinh\frac{1}{2}x = \sinh\frac{1}{2}\alpha \cosh y$ we have

$$I = \frac{\pi^2}{2\beta_0^3} \int_0^\infty \frac{\mathrm{d}y}{\cosh\frac{1}{2}x} \frac{\sinh(\pi x/2\beta_0)}{\cosh^3(\pi x/2\beta_0)}.$$

$I$ is readily obtained when $\pi/\beta_0$ is a positive integer $N$:

$N = 1$ (flat disc)

$$I = \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}y}{\cosh\frac{1}{2}x} = \frac{1}{8}\frac{1}{\cosh^3\frac{1}{2}\alpha} \qquad \frac{E_{\mathscr{S}}}{E_0} = 1$$

$N = 2$ (hemisphere)

$$I = \frac{3}{2\sqrt{2}} \frac{1}{(2\cosh^2\frac{1}{2}\alpha - 1)^{5/2}} \qquad \frac{E_{\mathscr{S}}}{E_0} = \frac{3}{\cosh\alpha} = 3\cos\theta$$

$N = 3$ ($\beta_0 = \pi/3$)

$$I = \frac{1}{8}\left(\frac{1}{\cosh^3\frac{1}{2}\alpha} + \frac{36}{(4\cosh^2\frac{1}{2}\alpha - 3)^{5/2}} - \frac{8}{(4\cosh^2\frac{1}{2}\alpha - 3)^{3/2}}\right)$$

$$\frac{E_{\mathcal{S}}}{E_0}=\frac{\sqrt{2}}{4}\left(\frac{1}{1+\cos\theta}\right)^{3/2}+3\cos\theta+\tfrac{1}{2}\qquad \text{with } E_S/E_0=\tfrac{29}{8}$$

$N=4$ $(\beta_0=\pi/4)$

$$I=\frac{1}{8}\left[\sqrt{2}\left(\frac{1}{(\cosh^2\frac{1}{2}\alpha-b)^{3/2}}-\frac{1}{(\cosh^2\frac{1}{2}\alpha-a)^{3/2}}\right)+\frac{3}{4}\left(\frac{1}{(\cosh^2\frac{1}{2}\alpha-a)^{5/2}}+\frac{1}{(\cosh^2\frac{1}{2}\alpha-b)^{5/2}}\right)\right]$$

with

$$a=\tfrac{1}{2}(1+1/\sqrt{2})\qquad b=\tfrac{1}{2}(1-1/\sqrt{2})$$

$$\frac{E_{\mathcal{S}}}{E_0}=\sqrt{2}\left[\left(\frac{\sqrt{2}}{4\cos\theta+3\sqrt{2}}\right)^{3/2}-1\right]+3\left(\cos\theta+\frac{1}{\sqrt{2}}\right)\left[1+\left(\frac{\sqrt{2}}{4\cos\theta+3\sqrt{2}}\right)^{5/2}\right]$$

with

$$E_S/E_0=19\sqrt{2}-23=3.870.$$

## 4. Electrical field on the axis

### 4.1. *Expression of* $E(\text{axis})/E_0$

Knowing $V$, it is easy to find that

$$\frac{E(\text{axis})}{E_0}=(2-2\cos\beta)\left(\frac{\sin\beta}{(2-2\cos\beta)^{1/2}}\int_0^\infty \tau\tanh\pi\tau\frac{\sinh(\beta_0-\beta)\tau}{\sinh\beta_0\tau}\,d\tau-(2-2\cos\beta)^{1/2}\int_0^\infty \tau^2\tanh\pi\tau\frac{\cosh(\beta_0-\beta)\tau}{\sinh\beta_0\tau}\,d\tau\right)$$

or

$$\frac{E(\text{axis})}{E_0}=1+4\sin^2\frac{\beta}{2}\cos\frac{\beta}{2}\int_0^\infty \tau\frac{\sinh(\pi-\beta_0)\tau}{\cosh\pi\tau}\frac{\sinh\tau\beta}{\sinh\tau\beta_0}\,d\tau+8\sin^3\frac{\beta}{2}\int_0^\infty \tau^2\frac{\sinh(\pi-\beta_0)\tau}{\cosh\pi\tau}\frac{\cosh\tau\beta}{\sinh\tau\beta_0}\,d\tau.$$

A numerical computation of these two integrals can be made, either using Gauss–Laguerre's method (Rabinowitz and Weiss 1959) or by replacing integrals by Riemann functions (Campbell 1966).

### 4.2. $\pi/\beta_0=N$ *(N positive integer)*

In this case

$$\frac{E(\text{axis})}{E_0}=1+2\frac{\beta}{\beta_0^3}\sin^2\frac{\beta}{2}\left(\cos\frac{\beta}{2}\right)S_1+\frac{2}{\beta_0^3}\left(\sin^3\frac{\beta}{2}\right)S_2$$

with

$$S_1 = \sum_{n=0}^{\infty} (-1)^n \sum_{m=1}^{N-1} \frac{m+Nn}{[(m+Nn)^2 - y^2]^2}$$

$$S_2 = \sum_{n=0}^{\infty} (-1)^n \sum_{m=1}^{N-1} \left(\frac{1}{[(m+Nn)-y]^3} + \frac{1}{[(m+Nn)+y]^3}\right)$$

and

$$y = \beta/2\beta_0.$$

*4.3. Check of the above result for a hemisphere*

$$S_1 = \sum_{n=0}^{\infty} (-1)^n \frac{1+2n}{[(1+2n)^2 - y^2]^2} = \frac{\pi^2}{16y} \frac{\sin \frac{1}{2}\pi y}{\cos^2 \frac{1}{2}\pi y}$$

$$S_2 = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{[(1+2n)-y]^3} + \frac{1}{[(1+2n)+y]^3}\right) = \frac{\pi^3}{16}\left(\frac{2}{\cos^3 \frac{1}{2}\pi y} - \frac{1}{\cos \frac{1}{2}\pi y}\right)$$

with $y = \beta/2\beta_0 = \beta/\pi$. After reduction,

$$E(\text{axis})/E_0 = 1 + 2\tan^3 \tfrac{1}{2}\beta = 1 + 2(c/z)^3$$

which is the well known formula.

*4.4. Limiting case: sphere*

According to the above results,

$$S_1 = \sum_{n=1}^{\infty} \frac{n}{(n^2-y^2)^2} + \sum_{n=1}^{\infty} (-1)^n \frac{nv}{[(nv)^2 - y^2]^2} + 2 \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} (-1)^n \frac{m+nv}{[(m+nv)^2 - y^2]^2}$$

with $v = \pi/\beta_0$. Then

$$\lim_{v\to\infty} S_1 = \sum_{n=1}^{\infty} \frac{n}{(n^2-y^2)^2}$$

and in the same way

$$\lim_{v\to\infty} S_2 = 2 \sum_{n=1}^{\infty} n \frac{n^2+3y^2}{(n^2-y^2)^3}$$

and

$$\frac{E(\text{axis})}{E_0} = 1 + 8y^3 \sum_{n=1}^{\infty} n \frac{n^2+y^2}{(n^2-y^2)^3} \qquad \text{with } y = \frac{a}{z}.$$

We find again a classical result (Boulloud 1958), which can be written with very good accuracy:

$$\frac{E(\text{axis})}{E_0} = 1 + 8y^3\left(\frac{1+y^2}{(1-y^2)^3} + 2\frac{4+y^2}{(4-y^2)^3} + 3\frac{9+y^2}{(9-y^2)^3} + A(1) + 2^2y^2A(2) + 3^2y^4A(3) + \ldots\right)$$

with

$$A(1) = 0.400\,198\,662 \times 10^{-1} \qquad A(2) = 0.156\,252\,881 \times 10^{-2}$$
$$A(3) = 0.795\,300\,112 \times 10^{-4} \qquad A(4) = 0.446\,256\,266 \times 10^{-5}$$
$$A(5) = 0.262\,324\,850 \times 10^{-6} \qquad A(6) = 0.158\,096\,041 \times 10^{-7}.$$

## 5. Total electrical charge $Q$ on the protruding part of the sphere

### 5.1. $Q$ function

The $Q$ function is given by the expression

$$Q = -\varepsilon_0 \iint \left(\frac{\partial V}{\partial n}\right) \mathrm{d}S$$

with

$$\frac{\partial V}{\partial n} = \frac{\cosh\alpha - \cos\beta}{c}\frac{\partial V}{\partial \beta} \qquad \mathrm{d}S = 2\pi c^2 \frac{(\sinh\alpha)\,\mathrm{d}\alpha}{(\cosh\alpha - \cos\beta_0)^2}.$$

Hence

$$Q = 8\pi\varepsilon_0 E_0 c^2 \int_0^\infty \frac{(\sinh\alpha)\,\mathrm{d}\alpha}{(2\cosh\alpha - 2\cos\beta_0)^{1/2}} \int_0^\infty \tau^2 \frac{\tanh\pi\tau}{\sinh\beta_0\tau} P_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau.$$

Reversing the order of integration, we have

$$Q = 8\pi\varepsilon_0 E_0 c^2 \int_0^\infty \tau^2 \frac{\tanh\pi\tau}{\sinh\beta_0\tau}\,\mathrm{d}\tau \int_0^\infty \frac{(\sinh\alpha) P_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\alpha}{(2\cosh\alpha - 2\cos\beta_0)^{1/2}}.$$

But

$$\int_0^\infty \frac{(\sinh\alpha) P_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\alpha}{(2\cosh\alpha - 2\cos\beta_0)^{1/2}}$$
$$= \int_0^\infty (\sinh\alpha) P_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\alpha \int_0^\infty \frac{\cosh(\pi-\beta_0)\tau}{\cosh\pi\tau} P_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau$$

and hence

$$Q = 8\pi\varepsilon_0 E_0 c^2 \int_0^\infty \tau^2 \frac{\tanh\pi\tau}{\sinh\beta_0\tau}\,\mathrm{d}\tau \int_0^\infty (\sinh\alpha) P_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\alpha$$
$$\times \int_0^\infty \left(\frac{\cosh(\pi-\beta_0)\tau}{\cosh\pi\tau}\,\frac{1}{\tau\tanh\pi\tau}\right)(\tau\tanh\pi\tau) P_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau.$$

By using the Mehler–Fock transform (Lebedev *et al* 1965), we obtain

$$Q = 8\pi\varepsilon_0 E_0 c^2 \int_0^\infty \frac{\tau\cosh(\pi-\beta_0)\tau\,\mathrm{d}\tau}{\sinh\beta_0\tau\cosh\pi\tau} = 8\pi\varepsilon_0 E_0 c^2 \int_0^\infty \tau(\coth\beta_0\tau - \tanh\pi\tau)\,\mathrm{d}\tau$$

and after some simple calculations

$$Q = 4\pi\varepsilon_0 E_0 c^2 \left(\frac{1}{\beta_0^2}\frac{\pi^2}{6} + \frac{1}{12}\right) = 4\pi\varepsilon_0 E_0 c^2 k_{cQ}$$

or

$$Q = 4\pi\varepsilon_0 E_0 a^2 \sin^2 \beta_0 \left(\frac{1}{\beta_0^2}\frac{\pi^2}{6} + \frac{1}{12}\right) = 4\pi\varepsilon_0 E_0 a^2 k_{aQ}.$$

### *5.2. Particular cases*

The last general results can be checked for some particular values of $\beta_0$ ($\pi/\beta_0 = N$, $N = 1, 2, 3, \ldots$). However, when $N \geq 4$, analytical computations are time consuming.

### *5.3. Other results*

Let $h$ be the penetration depth of the sphere in the plane, and $y = h/2a$ (figure 1). Then $h = 2a \sin^2 \frac{1}{2}\beta_0$ and

$$k_{aQ} = \tfrac{1}{3}y(1-y)[1 + \tfrac{1}{2}(\pi/\sin^{-1}\sqrt{y})^2].$$

When $y$ is small

$$k_{aQ} = 1.645 - 1.85y + 0.10y^2$$

## 6. Force *F*

### *6.1. Expression of F*

The force acting on a portion of sphere is given by

$$F = \tfrac{1}{2}\varepsilon_0 \iint_{\mathscr{S}} \left(\frac{\partial V}{\partial n}\right)^2 \mathrm{d}S \cos\theta$$

with

$$\mathrm{d}S \cos\theta = 2\pi c^2 \frac{1 - \cos\beta_0 \cosh\alpha}{(\cosh\alpha - \cos\beta_0)^3} \sinh\alpha \, \mathrm{d}\alpha.$$

Then

$$F = 8\pi\varepsilon_0 E_0^2 c^2 \int_0^\infty (1 - \cos\beta_0 \cosh\alpha) \sinh\alpha \times \left(\int_0^\infty \tau^2 \frac{\tanh \pi\tau}{\sinh \beta_0\tau} P_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau\right)^2 \mathrm{d}\alpha$$

or

$$F = 4\pi\varepsilon_0 E_0^2 c^2 k_{cF} = 4\pi\varepsilon_0 E_0^2 a^2 k_{aF}$$

with

$$k_{cF} = 2I_1 - 2(\cos\beta_0)I_2$$

$$k_{aF} = k_{cF} \sin^2\beta_0$$

$$I_1 = \int_0^\infty \sinh\alpha \left(\int_0^\infty \tau^2 \frac{\tanh \pi\tau}{\sinh \beta_0\tau} P_{-1/2+\mathrm{i}\tau}(\cosh\alpha)\,\mathrm{d}\tau\right)^2 \mathrm{d}\alpha$$

$$I_2=\int_0^\infty \cosh\alpha\,\sinh\alpha\left(\int_0^\infty \tau^2\frac{\tanh\pi\tau}{\sinh\beta_0\tau}P_{-1/2+i\tau}(\cosh\alpha)\,d\tau\right)^2 d\alpha.$$

### 6.2. Particular cases

$I_1$, $I_2$ and $k_F$ can be readily computed when $I$ is known.

$\beta_0=\pi$ (flat disc)

$$I=\frac{1}{8}\frac{1}{\cosh^3\frac{1}{2}\alpha}\qquad I_1=\tfrac{1}{64}\qquad I_2=\tfrac{3}{64}\qquad k_{cF}=\tfrac{1}{8}$$

$\beta_0=\frac{1}{2}\pi$ (hemisphere)

$$I=\frac{3}{2\sqrt{2}}\frac{1}{(2\cosh^2\frac{1}{2}\alpha-1)^{5/2}}\qquad I_1=\tfrac{9}{32}\qquad I_2=\tfrac{3}{8}\qquad k_{cF}=\tfrac{9}{16}=k_{aF}$$

$\beta_0=\frac{1}{3}\pi$

$$k_{cF}=2I_1-2(\cos\beta)I_2=\tfrac{281}{3}\times\tfrac{1}{32}-\tfrac{5}{3}=1.260\qquad k_{aF}=0.945$$

$\beta_0=\frac{1}{4}\pi$

After some computations

$$k_{cF}=\tfrac{1}{8}(131-80\sqrt{2})=2.232\qquad k_{aF}=1.116.$$

## 7. Electrical field on the plane

Knowing $V$ and

$$E=-\frac{\partial V}{\partial n}=-\frac{\cosh\alpha-\cos\beta}{c}\frac{\partial V}{\partial\beta},$$

we obtain

$$\frac{E(\beta=0)}{E_0}=(2\cosh\alpha-2)^{3/2}\int_0^\infty \tau^2\frac{\tanh\pi\tau}{\tanh\beta_0\tau}P_{-1/2+i\tau}(\cosh\alpha)\,d\tau.$$

By using the integral representation of Mehler's function and reversing the order of integration,

$$\frac{E(\beta=0)}{E_0}=\frac{2}{\pi}(2\cosh\alpha-2)^{3/2}\lim_{x\to\alpha}\int_x^\infty\frac{dx}{(2\cosh x-2\cosh\alpha)^{1/2}}\int_0^\infty\frac{\tau^2\sin x\tau\,d\tau}{\tanh\beta_0\tau}$$

and after simplification

$$\frac{E(\beta=0)}{E_0}=\frac{\pi^2}{\beta_0^3}\left(8\sinh^3\frac{\alpha}{2}\right)\lim_{x\to\alpha}\int_\alpha^\infty\frac{\cosh(\pi x/2\beta_0)\frac{1}{2}\,dx}{(2\cosh x-2\cosh\alpha)^{1/2}\sinh^3(\pi x/2\beta_0)}.$$

When $\pi/\beta_0=N$, the integral can be easily computed.

**Table 1.** Charge and force coefficients versus $\beta_0$ in degrees.

| $\beta_0$ | $k_{aQ}$ | $k_{aF}$ |
|---|---|---|
| 0 | 1.644 937 | 1.368 724 |
| 1 | 1.644 792 | 1.368 590 |
| 2 | 1.644 368 | 1.368 188 |
| 3 | 1.643 660 | 1.367 518 |
| 4 | 1.642 669 | 1.366 580 |
| 5 | 1.641 396 | 1.365 376 |
| 6 | 1.639 840 | 1.363 905 |
| 7 | 1.638 004 | 1.362 168 |
| 8 | 1.635 886 | 1.360 165 |
| 9 | 1.633 489 | 1.357 899 |
| 10 | 1.630 812 | 1.355 368 |

**Table 2.** Electrical field, charge and force coefficients versus $\beta_0$ in degrees.

| $\beta_0$ | $E_S/E_0$ | $k_{aQ}$ | $k_{aF}$ |
|---|---|---|---|
| 0 | 4.2072 | 1.6449 | 1.3687 |
| 5 | 4.2029 | 1.6414 | 1.3654 |
| 10 | 4.1899 | 1.6308 | 1.3554 |
| 15 | 4.1685 | 1.6133 | 1.3388 |
| 20 | 4.1386 | 1.5889 | 1.3159 |
| 25 | 4.1004 | 1.5580 | 1.2868 |
| 30 | 4.0542 | 1.5208 | 1.2519 |
| 35 | 4.0002 | 1.4777 | 1.2116 |
| 40 | 3.9387 | 1.4289 | 1.1663 |
| 45 | 3.8701 | 1.3750 | 1.1164 |
| 50 | 3.7946 | 1.3164 | 1.0626 |
| 55 | 3.7128 | 1.2538 | 1.0054 |
| 60 | 3.6250 | 1.1875 | 0.9453 |
| 65 | 3.5317 | 1.1183 | 0.8831 |
| 70 | 3.4334 | 1.0467 | 0.8194 |
| 75 | 3.3306 | 0.9734 | 0.7548 |
| 80 | 3.2237 | 0.8991 | 0.6901 |
| 85 | 3.1133 | 0.8244 | 0.6257 |
| 90 | 3.0000 | 0.7500 | 0.5625 |
| 95 | 2.8842 | 0.6765 | 0.5009 |
| 100 | 2.7665 | 0.6045 | 0.4416 |
| 105 | 2.6474 | 0.5347 | 0.3849 |
| 110 | 2.5275 | 0.4677 | 0.3315 |
| 115 | 2.4071 | 0.4038 | 0.2815 |
| 120 | 2.2869 | 0.3437 | 0.2355 |
| 125 | 2.1673 | 0.2878 | 0.1935 |
| 130 | 2.0487 | 0.2364 | 0.1558 |
| 135 | 1.9315 | 0.1898 | 0.1225 |
| 140 | 1.8162 | 0.1483 | 0.0936 |
| 145 | 1.7032 | 0.1119 | 0.0690 |
| 150 | 1.5927 | 0.0808 | 0.0486 |
| 155 | 1.4852 | 0.0550 | 0.0322 |
| 160 | 1.3808 | 0.0344 | 0.0196 |
| 165 | 1.2799 | 0.0189 | 0.0104 |
| 170 | 1.1827 | 0.0081 | 0.0044 |
| 175 | 1.0893 | 0.0020 | 0.0010 |
| 180 | 1.0000 | 0.0000 | 0.0000 |

## 8. Numerical results

When $\beta_0$ (in degrees) is small, we obtain the values of $k_{aQ}$ and $k_{aF}$ listed in table 1. These results allow an approximation of $k_{aF}$ in the neighbourhood of $\beta_0 = 0$:

$$k_{aF}(\beta_0) = k_{aF}(0)\left[1 - \frac{9}{28}\left(\frac{\pi}{180}\right)^2 \beta_0^2 + \frac{1}{27}\left(\frac{\pi}{180}\right)^4 \beta_0^4\right].$$

This result is very useful for computation of adhesion forces (St John 1971).

Table 2 gives $E_S/E_0$, $k_{aQ}$ and $k_{aF}$. Intermediate values are easily obtained by classical interpolation formulae (Mineur 1966). This method also allows the calculation of the numerical values of these coefficients as functions of $h/2a$, the penetration coefficient of the sphere in the plane.

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